Final - Partial Differential Equations (2022-23)
Time: 3 hours.
Attempt all questions, giving proper explanations.

1. Let $H : \mathbf{R}^n \to \mathbf{R}$ be convex. We say $\mathbf{q}$ belongs to the subdifferential of H at $\mathbf{p}$, written $\mathbf{q} \in \partial H(\mathbf{p})$, if
$$H(\mathbf{r}) \geq H(\mathbf{p}) + \mathbf{q} \cdot (\mathbf{r} - \mathbf{p}), \qquad \text{for all } \mathbf{r} \in \mathbf{R}^n.$$
Prove $\mathbf{q} \in \partial H(\mathbf{p})$ if and only if $\mathbf{p} \in \partial L(\mathbf{q})$ if and only if $\mathbf{p} \cdot \mathbf{q} = H(\mathbf{p}) + L(\mathbf{q})$, where $L = H^*$. **[6 marks]**

2. Let $L : \mathbf{R}^n \to \mathbf{R}$ be smooth, convex with $\lim_{|q|\to\infty} L(q)/|q| = \infty$, and $g : \mathbf{R}^n \to \mathbf{R}$ be Lipschitz. For $(\mathbf{x}, t) \in \mathbf{R}^n \times (0, \infty)$ let
$$u(\mathbf{x}, t) = \min_{\mathbf{y} \in \mathbf{R}^n} \left\{ tL\left(\frac{\mathbf{x} - \mathbf{y}}{t}\right) + g(\mathbf{y}) \right\}.$$
Show that for $0 < s < t$
$$u(\mathbf{x}, t) \geq \min_{\mathbf{y} \in \mathbf{R}^n} \left\{ (t - s)L\left(\frac{\mathbf{x} - \mathbf{y}}{t - s}\right) + u(\mathbf{y}, s) \right\}. \qquad \textbf{[6 marks]}$$

3. Fix $\mu > 0$ and let ϕ be a bounded continuous function on $(0, \infty)$. Solve the following heat equation on the half line $(0, \infty)$. **[8 marks]**
$$\begin{aligned} v_t - \mu v_{xx} &= 0 && \text{in } \{0 < x < \infty,\ 0 < t < \infty\}, \\ v(x, 0) &= \phi(x) && \text{for } x > 0, \\ v(0, t) &= 0 && \text{for } t > 0. \end{aligned}$$

4. Let $g : [0, 1] \to \mathbf{R}$ be continuous and let u be a solution to
$$\begin{aligned} u_t(x, t) &= u_{xx}(x, t), && 0 < x < 1,\ t > 0, \\ u(x, 0) &= g(x) && 0 \leq x \leq 1, \\ u_x(0, t) &= u_x(1, t) = 0, && t > 0. \end{aligned}$$
Let $U = \int_0^1 g(x)dx$, and let $w(x, t) = u(x, t) - U$.
   (a) Show that $\int_0^1 w(x, t)dx = 0$ for all $t > 0$. **[2 marks]**
   (b) Show that there is an $x(t)$ such that $w(x(t), t) = 0$, and conclude that for any $x \in [0, 1]$
$$w(x, t) = \int_{x(t)}^{x} w_x(y, t)dy. \qquad \textbf{[2 marks]}$$
   (c) Show that
$$\int_0^1 w^2(y, t)dy \leq \int_0^1 w_x^2(y, t)dy. \qquad \textbf{[2 marks]}$$
   (d) Let $E(t) = \int_0^1 w^2(x, t)dx$. Show that
$$E'(t) = -2\int_0^1 w_x^2(x, t)dx. \qquad \textbf{[2 marks]}$$
   (e) Show that $E(t) \to 0$ as $t \to \infty$. **[4 marks]**

5. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,\ u(x, 0) = x^2,\ u_t(x, 0) = e^x$. **[6 marks]**
(HINT: Factor the operator)

6. Let $\phi(\mathbf{x})$ be any C^2 function defined on $\mathbf{R}^3$ that vanishes outside some ball $B(\mathbf{0}, R)$. Show that

$$\phi(\mathbf{0}) = -\int_{B(\mathbf{0},2R)} \frac{\Delta\phi(\mathbf{x})}{4\pi|\mathbf{x}|} d\mathbf{x}. \qquad \textbf{[8 marks]}$$

7. Argue how the Lax-Oleinik solution to the scalar conservation law

$$\begin{aligned} u_t + F(u)_x &= 0 \quad \text{in } \mathbf{R} \times (0, \infty), \\ u &= g \quad \text{on } \mathbf{R} \times \{t = 0\}, \end{aligned}$$

where F is uniformly convex and $g \in L^\infty(\mathbf{R})$, follows from the Hopf-Lax formula for the Hamilton-Jacobi equation. **[4 marks]**